

SOLUTIONS AND STABILITY OF VARIANT OF WILSON'S FUNCTIONAL EQUATION

ELQORACHI ELHOUCIEN AND REDOUANI AHMED

ABSTRACT. In this paper we will investigate the solutions and stability of the generalized variant of Wilson's functional equation

$$(E) : \quad f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in G,$$

where G is a group, σ is an involutive morphism of G and χ is a character of G . (a) We solve (E) when σ is an involutive automorphism, and we obtain some properties about solutions of (E) when σ is an involutive anti-automorphism. (b) We obtain the Hyers Ulam stability of equation (E) . As an application, we prove the superstability of the functional equation $f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)f(y)$, $x, y \in G$.

1. INTRODUCTION

D'Alembert's functional equation

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G$$

also called the cosine functional equation has a long history going back to J.d'Alembert. Equation (1.1) plays an important role in determining the sum of two vectors in various Euclidean and non-Euclidean geometries.

The continuous solutions $f: \mathbb{R} \rightarrow \mathbb{C}$ of d'Alembert's functional equation (1.1) are known: A part from the trivial solution $f = 0$, they are $f_\lambda(x) = \cos(\lambda x)$, $x \in \mathbb{R}$ where the parameter λ ranges over \mathbb{C} (see for example [1])

Several authors have determined the general solution $f: G \rightarrow \mathbb{C}$ of the following generalization of d'Alembert's functional equation

$$(1.2) \quad f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G$$

in abelian case and in non abelian case.

Probably the first result in non abelian group was obtained by Kannappan [27]. Under the condition $f(xyz) = f(yxz)$ for all $x, y, z \in G$, the solutions of equation (1.2) are of the form $f(x) = \frac{\phi(x) + \phi(\sigma(x))}{2}$, where ϕ is multiplicative. There has been quite a development of the theory of d'Alembert's functional equation (1.1) during the last years, on non abelian groups, as shown in works by Dilian yang about compact groups [10, 11, 12], Stetkær [42] for step 2 nilpotent groups, Friis [17] for results on Lie groups and Davison [8, 9]

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for general groups, even monoids. The most comprehensive recent study is by Stetkær [41, 37].

Recently, Stetkær [39] obtained the complex valued solutions of the following version of d'Alembert's functional equation

$$(1.3) \quad f(xy) + \chi(y)f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G,$$

where $\chi : G \rightarrow \mathbb{C}$ is a character of G . The non-zero solutions of equation (1.3) are the normalized traces of certain representation of the group G on \mathbb{C}^2

In the same year Stetkær [38] obtained the complex valued solution of the following variant of d'Alembert's functional equation

$$(1.4) \quad f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G,$$

where σ is an involutive homomorphism of G . The solutions of equation (1.4) are of the form $f(x) = \frac{\varphi(x) + \varphi(\sigma(x))}{2}$, $x \in G$, where φ is multiplicative. In [13] Ebanks and Stetkær studied the solutions $f, g: G \rightarrow \mathbb{C}$ of Wilson's functional equation

$$(1.5) \quad f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G$$

and the following variant of Wilson's functional equation (see [44])

$$(1.6) \quad f(xy) + f(y^{-1}x) = 2f(x)g(y), \quad x, y \in G.$$

They solve (1.6) and they obtained some new results about (1.5). We refer also to Wilson's first generalization of d'Alembert's functional equation:

$$(1.7) \quad f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in \mathbb{R}.$$

For more about the functional equation (1.7) see Aczél [[1], Section 3.2.1 and 3.2.2]. The solutions formulas of equation (1.7) for abelian groups are known.

The stability of d'Alembert's functional equation was first obtained by Baker [5] when the following theorem was proved.

Theorem 1.1. [5] (*Superstability of d'Alembert's functional equation*) *Let G be a group. If a function $f: G \rightarrow \mathbb{C}$ satisfies the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in G$, then either f is bounded on G or $f(x+y) + f(x-y) = 2f(x)f(y)$ for all $x, y \in G$.

A different generalization of Baker's result was given by L. Székelyhidi [46, 47, 48]. It involves an interesting generalization of the class of bounded function on a group or semigroup. For other stability and superstability results, we can see for example [2], [3], [4], [7], [14], [19], [20] and [36], the present authors [6] for general groups.

Various stability results of Wilson's functional equation and its generalization are obtained. The number of papers in this subject is very high, hence, it is not realistic to try to refer to all. The interested reader should refer to

[16], [18], [15], [21]-[35] for a thorough account on the subject of stability of functional equations.

The main purpose of this paper is to study the solutions and stability of the more general variant of Wilson's functional equation

$$(1.8) \quad f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in G,$$

where G is a group, χ is a character of G , σ is an involutive morphism of G . That is, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$ or $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$.

We solve (1.8) when σ is an involutive automorphism, and we obtain some properties of the solutions of equation (1.8) when σ is an involutive anti-automorphism. Furthermore, we obtain the Hyers Ulam stability of equation (1.8). As an application we prove the superstability of the functional equation

$$(1.9) \quad f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G.$$

2. STABILITY OF THE FUNCTIONAL EQUATION (1.8), WHERE σ IS AN INVOLUTIVE ANTI-AUTOMORPHISM OF G .

In this section σ is an involutive anti-automorphism of G , that is $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$, for all $x, y \in G$. The following theorem is one of the main results of the present paper.

Theorem 2.1. *Let $\delta \geq 0$. Let σ be an involutive anti-automorphism of G . Let χ be a unitary character of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the functions $f, g: G \rightarrow \mathbb{C}$ satisfy the inequality*

$$(2.1) \quad |f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)| \leq \delta$$

for all $x, y \in G$. Under these assumptions the following statements hold:

(1) *If f is unbounded, then*

(i) *g is central. That is $g(xy) = g(yx)$, for all $x, y \in G$; $m_g: G \rightarrow \mathbb{C}^*$ is multiplicative.*

(ii) *$g(x) = \chi(x)g(\sigma(x))$ for all $x \in G$, (if $\sigma(x) = x^{-1}$, $\check{\chi}m_g(G) \subseteq \{\mp 1\}$).*

$$(iii) \quad g(x) = m_g(x)g(x^{-1}) \text{ for all } x \in G$$

and

(vi)

$$(2.2) \quad g(xy) + m_g(y)g(xy^{-1}) = 2g(x)g(y) \text{ for all } x, y \in G,$$

where $m_g(x) = 2g(x)^2 - g(x^2)$, $x \in G$.

(2) *If g is unbounded and $f \neq 0$, then*

(v) *the pair (f, g) satisfies the functional equation (1.8). Furthermore,*

(vi) *$m_g: G \rightarrow \mathbb{C}^*$ is multiplicative, (if $\sigma(x) = x^{-1}$, $\check{\chi}m_g(G) \subseteq \{\mp 1\}$).*

(vii) *$g(x) = m_g(x)g(x^{-1})$, $g(x) = \chi(x)g(\sigma(x))$, $\chi(y)f(\sigma(y)xy) = m_g(y)f(x)$*

for all $x, y \in G$.

(viii) The pair (f, g) satisfies

$$(2.3) \quad f(xy) + m_g(y)f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G$$

and g satisfies equation (2.2).

Proof. All technical methods that are needed in our discussion are due to Stetkær [44]. (1) We let L and R denote respectively: the left and right regular representation of G on functions on G . That is, $[L(y)h](x) = h(\sigma(y)x)$ and $R(y)h(x) = h(xy)$ for $x, y \in G$ and $h: G \rightarrow \mathbb{C}$. We notice here that $L(x)R(y) = R(y)L(x)$ and $L(yz)h = L(y)[L(z)h]$, $R(yz)h = R(y)[R(z)h]$ for all $x, y \in G$ and for all function $h: G \rightarrow \mathbb{C}$.

Thus, inequality (2.1) can be written as follows

$$(2.4) \quad \|[R(y) + \chi(y)L(y)]f - 2g(y)f\|_\infty \leq \delta$$

for all $y \in G$. Applying the bounded operator $R(z) + \chi(z)L(z)$ to the bounded function $R(y) + \chi(y)L(y)f - 2g(y)f$ we get after reduction that

$$(2.5) \quad \begin{aligned} & \|(R(z) + \chi(z)L(z))[R(y) + \chi(y)L(y)]f - 2g(y)(R(z) + \chi(z)L(z))f\|_\infty \\ &= \|[(R(z)y) + \chi(z)y)L(z)y)f - 2g(z)yf + 2g(z)yf + \chi(y)R(z)L(z) + \chi(z)L(z)R(y)]f \\ & \quad - 2g(y)(R(z) + \chi(z)L(z))f - 2g(z)f - 4g(z)g(y)f\|_\infty. \end{aligned}$$

By using (2.4), $\|(R(z) + \chi(z)L(z))h\|_\infty \leq 2\|h\|_\infty$ for all complex bounded function h on G we obtain

$$(2.6) \quad \|2g(z)yf + [\chi(y)R(z)L(z) + \chi(z)L(z)R(y)]f - 4g(z)g(y)f\|_\infty \leq 3\delta + 2|g(y)|\delta$$

for all $z, y \in G$.

(1) (i) Interchanging z and y in (2.6) and subtracting the result obtained from (2.6) we get

$$\|[g(zy) - g(yz)]f\|_\infty \leq 2|g(z)|\delta + 2|g(y)|\delta + 6\delta.$$

Since f is assumed to be unbounded, then g is central.

Setting $y = z$ in (2.6), we obtain

$$(2.7) \quad \|(2g(y)^2 - g(y^2))f - \chi(y)R(y)L(y)f\|_\infty \leq |g(y)|\delta + \frac{3}{2}\delta.$$

That is,

$$(2.8) \quad |(2g(y)^2 - g(y^2))f(x) - \chi(y)f(\sigma(y)xy)| \leq |g(y)|\delta + \frac{3}{2}\delta$$

for all $x, y \in G$. Which implies that

$$(2.9) \quad \|m_g(y)f - \chi(y)\mu(y)f\|_\infty \leq 2|g(y)|\delta + \frac{1}{2}\delta,$$

where $[\mu(y)h](x) = h(\sigma(y)xy)$. Noting that $\mu(yz) = \mu(y)\mu(z)$ for all $z, y \in G$. By using inequality (2.8) we have

$$|m_g(yz)f(x) - \chi(y)\chi(z)f(\sigma(z)\sigma(y)xyz)| \leq |g(yz)|\delta + \frac{3}{2}\delta,$$

$$|m_g(y)f(x) - \chi(y)f(\sigma(y)xy)| \leq |g(y)|\delta + \frac{3}{2}\delta,$$

and

$$|m_g(z)f(\sigma(y)xy) - \chi(z)f(\sigma(z)\sigma(y)xyz)| \leq |g(z)|\delta + \frac{3}{2}\delta.$$

So, by using triangle inequality we get

$$\begin{aligned} |m_g(yz)f(x) - m_g(y)m_g(z)f(x)| &\leq |m_g(yz)f(x) - \chi(y)\chi(z)f(\sigma(z)\sigma(y)xyz)| \\ &\quad + |\chi(y)\chi(z)f(\sigma(z)\sigma(y)xyz) - m_g(y)m_g(z)f(x)| \\ &\leq |m_g(yz)f(x) - \chi(y)\chi(z)f(\sigma(z)\sigma(y)xyz)| \\ &\quad + |m_g(z)\chi(y)f(\sigma(y)xy) - m_g(y)m_g(z)f(x)| + |\chi(y)\chi(z)f(\sigma(z)\sigma(y)xyz) - m_g(z)\chi(y)f(\sigma(y)xy)| \\ &\leq |g(yz)|\delta + \frac{3}{2}\delta + |m_g(z)||\chi(y)f(\sigma(y)xy) - m_g(y)f(x)| \\ &\quad + |\chi(y)||\chi(z)f(\sigma(z)\sigma(y)xyz) - m_g(z)f(\sigma(y)xy)| \\ &\leq |g(yz)|\delta + \frac{3}{2}\delta + |m_g(z)|[|g(y)|\delta + \frac{3}{2}\delta] + |\chi(y)|[|g(z)|\delta + \frac{3}{2}\delta]. \end{aligned}$$

From the assumption that f is unbounded we get $m_g(yz) = m_g(y)m_g(z)$ for all $y, z \in G$. On the other hand if $m_g = 0$, then if we put $y = e$ in (2.8) we obtain f bounded, since f is unbounded, so $m_g(G) \subseteq \mathbb{C}^*$.

(ii) Now, let $a \in G$ be arbitrary. First case: Assume that either $f(a) \neq 0$ or $f(e) \neq 0$. The pair (f, g) satisfies inequality (2.1) on the abelian subgroup $\langle a \rangle$ generated by a , then on the abelian subgroup $\langle a \rangle$ we have $|f(x\sigma(y)) + \chi(\sigma(y))f(xy) - 2f(x)g(\sigma(y))| \leq \delta$, since χ is unitary and $\chi(y\sigma(y)) = 1$ hence we get

$$(2.10) \quad |\chi(y)f(x\sigma(y)) + f(xy) - 2f(x)\chi(y)g(\sigma(y))| \leq \delta, \quad x, y \in G.$$

By substituting (2.1) to (2.10) on the commutative subgroup $\langle a \rangle$ we obtain $|f(x)[g(y) - \chi(y)g(\sigma(y))]| \leq 2\delta$ for all $x, y \in G$. Since f is unbounded, then we have $g(y) = \chi(y)g(\sigma(y))$ for all $y \in \langle a \rangle$. In particular $g(a) = \chi(a)g(\sigma(a))$

Second case: Assume that $f(a) = 0$ and $f(e) = 0$. Setting $x = e$ in (2.1), we obtain

$$(2.11) \quad |f(y) + \chi(y)f(\sigma(y))| \leq \delta$$

for all $y \in G$. Thus, from $\chi(a\sigma(a)) = 1$, $2f(x)[g(a) - \chi(a)g(\sigma(a))]$ can be written as follows

$$\begin{aligned} 2f(x)[g(a) - \chi(a)g(\sigma(a))] &= 2f(x)g(a) - f(xa) - \chi(a)f(\sigma(a)x) \\ &\quad + \chi(a)[f(x\sigma(a)) + \chi(\sigma(a))f(ax) - 2f(x)g(\sigma(a))] \\ &\quad + f(xa) + \chi(a)f(\sigma(a)x) - \chi(a)f(x\sigma(a)) - f(ax). \end{aligned}$$

Since

$$\begin{aligned} &f(xa) + \chi(a)f(\sigma(a)x) - \chi(a)f(x\sigma(a)) - f(ax) \\ &= f(xa) + \chi(a)f(\sigma(a)x) - \chi(a)[f(x\sigma(a)) + \chi(x\sigma(a))f(a\sigma(x))] + \chi(x)f(a\sigma(x)) \\ &\quad - [f(ax) + \chi(ax)f(\sigma(x)\sigma(a))] + \chi(ax)f(\sigma(x)\sigma(a)) \\ &= \chi(x)[f(a\sigma(x)) + \chi(\sigma(x))f(\sigma(\sigma(x))a) - 2f(a)g(\sigma(x))] + 2\chi(x)f(a)g(\sigma(x)) \end{aligned}$$

$$+ \chi(a)[f(\sigma(a)x) + \chi(x)f(\sigma(x)\sigma(a)) - 2f(\sigma(a))g(x)] + 2\chi(a)f(\sigma(a))g(x) \\ - \chi(a)[f(x\sigma(a)) + \chi(x\sigma(a))f(a\sigma(x))] - [f(ax) + \chi(ax)f(\sigma(x)\sigma(a))].$$

From inequalities (2.11), (2.1) and $|\chi(x)| = 1$ we get

$$(2.12) \quad |2f(x)[g(a) - \chi(a)g(\sigma(a))]| \leq 6\delta + 2|f(\sigma(a))|g(x) + 2|f(a)||g(\sigma(x))|$$

for all $x \in G$. Here again we discuss two subcases: If g is bounded, then by using the unboundedness of f and (2.12) we get $g(a) = \chi(a)g(\sigma(a))$. If g is unbounded we use the case *ii*) to obtain that $g(x) = \chi(x)g(\sigma(x))$ for all $x \in G$ hence we get the result for $x = a$. On the other hand we have $m_g(\sigma(y)) = 2g(\sigma(y))^2 - g(\sigma(y^2)) = 2\chi(y)^2g(y)^2 - \chi(\sigma(y))^2g(y^2) = \chi(y)^2m_g(y)$. So if $\sigma(x) = x^{-1}$ for all $x \in G$ then we get $m_g(x)m_g(x^{-1}) = m_g(e) = 1 = m_g(x)\chi(x^{-1})^2m_g(x) = (\chi(x^{-1})m_g(x))^2$, then we get $\check{\chi}m_g(G) \subseteq \{\pm 1\}$.

(iii) The formula $2f(x)[g(y) - m_g(y)\check{g}(y)]$ can be written as follows

$$2f(x)[g(y) - m_g(y)\check{g}(y)] = \\ [- (f(xy) + \chi(y)f(\sigma(y)x)) + 2f(x)g(y)] \\ + m_g(y)[f(xy^{-1}) + \chi(y^{-1})f(\sigma(y^{-1})x)) - 2f(x)\check{g}(y)] \\ + f(xy) - \chi(y^{-1})m_g(y)f(\sigma(y^{-1})x) + \chi(y)f(\sigma(y)x) - m_g(y)f(xy^{-1}).$$

Once again we have

$$f(xy) - \chi(y^{-1})m_g(y)f(\sigma(y^{-1})x) + \chi(y)f(\sigma(y)x) - m_g(y)f(xy^{-1}) \\ = f(\sigma(y)\sigma(y^{-1})xy) - \chi(y^{-1})m_g(y)f(\sigma(y^{-1})x) + \chi(y)f(\sigma(y)xy^{-1}) - m_g(y)f(xy^{-1}) \\ = [\mu(y)f](\sigma(y^{-1})x) - \chi(y^{-1})m_g(y)f(\sigma(y^{-1})x) + \chi(y)[\mu(y)f](xy^{-1}) - m_g(y)f(xy^{-1})$$

and from inequality (2.8) we obtain

$$(2.13) \quad |[\mu(y)f](\sigma(y^{-1})x) - \chi(y^{-1})m_g(y)f(\sigma(y^{-1})x) + \chi(y)[\mu(y)f](xy^{-1}) - m_g(y)f(xy^{-1})| \\ \leq 2|g(y)|\delta + 3\delta.$$

Now, from inequalities (2.1) and (2.13) we get

$$(2.14) \quad |2f(x)[g(y) - m_g(y)\check{g}(y)]| \leq |m_g(y)|\delta + 2|g(y)|\delta + 4\delta.$$

Since f is unbounded then we have $g(y) = m_g(y)\check{g}(y)$ for all $y \in G$.

(vi) Let us consider

$$(2.15) \quad 2f(x)[g(zy) + m_g(y)g(zy^{-1}) - 2g(y)g(z)]$$

$$= [2g(zy)f(x) - 4g(y)g(z)f(x) + \chi(y)(R(z)L(y)f)(x) + \chi(z)(L(z)R(y)f)(x)] \\ + 2f(x)m_g(y)g(zy^{-1}) - \chi(y)(R(z)L(y)f)(x) - \chi(z)(L(z)R(y)f)(x).$$

Since

$$2f(x)m_g(y)g(zy^{-1}) - \chi(y)(R(z)L(y)f)(x) - \chi(z)(L(z)R(y)f)(x) \\ = 2f(x)m_g(y)g(zy^{-1}) - \chi(y)f(\sigma(y)xzy^{-1}) - \chi(z)f(\sigma(y)\sigma(y^{-1})\sigma(z)xy) \\ = 2f(x)m_g(y)g(zy^{-1}) - \chi(y)f(\sigma(y)xzy^{-1}) + m_g(y)f(xzy^{-1}) - m_g(y)f(xzy^{-1}) \\ - \chi(z)f(\sigma(y)\sigma(y^{-1})\sigma(z)xy) + \chi(y^{-1})\chi(z)m_g(y)f(\sigma(y^{-1})\sigma(z)x)$$

$$\begin{aligned}
& -\chi(y^{-1})\chi(z)m_g(y)f(\sigma(y^{-1})\sigma(z)x)) \\
& = -\chi(y)[\mu(y)f](xzy^{-1}) + m_g(y)f(xzy^{-1}) \\
& -\chi(z)[\mu(y)f](\sigma(y^{-1})\sigma(z)x) + \chi(y^{-1})m_g(y)f(\sigma(y^{-1})\sigma(z)x)) \\
& -m_g(y)[f(xzy^{-1}) + \chi(zy^{-1})f(\sigma(y^{-1})\sigma(z)x) - 2f(x)g(zy^{-1})].
\end{aligned}$$

From inequalities (2.1), (2.8), (2.10) and (2.15) we obtain

$$|2f(x)[g(zy) + m_g(y)g(zy^{-1}) - 2g(y)g(z)]| \leq 6\delta + 4|g(y)|\delta + |m_g(y)|\delta.$$

Since f is unbounded, then g satisfies equation $g(xy) + m_g(y)g(xy^{-1}) = 2g(x)g(y)$, $x, y \in G$.

(2) We assume that g is unbounded and $f \neq 0$. By simple computations we get f unbounded. Now, for all $x, y, z \in G$ we have

$$\begin{aligned}
& |2g(z)||f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)| \\
& = |2f(xy)g(z) + 2\chi(y)f(\sigma(y)x)g(z) - 4f(x)g(y)g(z)| \\
& \leq |-f(xyz) - \chi(z)f(\sigma(z)xy) + 2f(xy)g(z)| \\
& + |\chi(y)[-f(\sigma(y)xz) - \chi(z)f(\sigma(z)\sigma(y)x) + 2f(\sigma(y)x)g(z)]| \\
& + |f(xyz) + \chi(yz)f(\sigma(z)\sigma(y)x) - 2f(x)g(yz)| \\
& + |-f(xzy) - \chi(zy)f(\sigma(y)\sigma(z)x) + 2f(x)g(zy)| \\
& + |\chi(z)f(\sigma(z)xy) + \chi(zy)f(\sigma(y)\sigma(z)x) - 2\chi(z)f(\sigma(z)x)g(y)| \\
& + |f(xzy) + \chi(y)f(\sigma(y)xz) - 2f(xz)g(y)| \\
& + 2|f(x)||g(yz) - g(zy)| + 2g(y)|f(xz) + \chi(z)f(\sigma(z)x) - 2f(x)g(z)|. \\
& \leq \delta + \delta + \delta + \delta + |\chi(z)|\delta + 0 \times 2|f(x)|\delta + 2|g(y)|\delta = 6\delta + 2|g(y)|\delta
\end{aligned}$$

Using that g is unbounded, we get the desired result that the pair (f, g) satisfies the functional equation (1.8). Now, by using (2.8) with $\delta = 0$ we get

$$(2.16) \quad m_g(y)f(x) = \chi(y)f(\sigma(y)xy)$$

for all $x, y \in G$. So if we replace x by xy^{-1} in (2.16) we obtain $m_g(y)f(xy^{-1}) = \chi(y)f(\sigma(y)x)$ and equation (1.8) can be written as follows $f(xy) + m_g(y)f(xy^{-1}) = 2f(x)g(y)$ for all $x, y \in G$. For the proof of other properties we use case (1) with $\delta = 0$. This completes the proof of theorem. \square

As an application we get some properties of the solutions of equation (1.8), where σ is an involutive anti-automorphism. By using the above Theorem for $\delta = 0$, [[13], Proposition 2, Theorem 3 and Corollary 6], [[39], Theorem 6.1, Theorem 10.1 and Corollary 10.2] we obtain the following theorem.

For later use, we recall (see for example [13]) that a function $f: G \rightarrow \mathbb{C}$ is said to be abelian, if $f(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}) = f(x_1x_2\dots x_n)$ for all $x_1, x_2, \dots, x_n \in G$, all permutations σ and all $n = 1, 2, \dots$.

Theorem 2.2. *Let the pair $f, g: G \longrightarrow \mathbb{C}$ be a solution of the variant (1.8) of Wilson's functional equation such that $f \neq 0$.*

(1) If f is a nonzero central function. Then,

(i) $f = f(e)g$, when g is non abelian.

(ii) When g is abelian g has the form $g = \frac{\psi + \chi\psi\circ\sigma}{2}$ where $\psi: G \longrightarrow \mathbb{C}^$ is a character. If $\psi \neq \chi\psi\circ\sigma$, then $f = \alpha(\psi - \chi\psi\circ\sigma)/2 + \beta(\psi + \chi\psi\circ\sigma)/2$ for some $\alpha, \beta \in \mathbb{C}$. If $\psi = \chi\psi\circ\sigma$, then $f = \psi a + \beta\psi$ for some additive map $a: G \longrightarrow \mathbb{C}$ and some $\beta \in \mathbb{C}$.*

(2) (i) $g(e) = 1$, g is central, and $g = \chi g \circ \sigma$, $g = m_g \check{g}$.

(ii) $m_g: G \longrightarrow \mathbb{C}^$ is multiplicative.*

(iii) $\chi(y)f(\sigma(y)xy) = m_g(y)f(x)$, $\chi(y)f(\sigma(y)x) = m_g(y)f(xy^{-1})$ for all $x, y \in G$.

(iv) $g(xy) + m_g(xy^{-1}) = 2g(x)g(y)$ for all $x, y \in G$.

(v) $f(xy) + m_g(y)f(xy^{-1}) = 2f(x)g(y)$ for all $x, y \in G$.

(vi) $f = -\chi f \circ \sigma$ if and only if $f(e) = 0$

(vii) The even part and the odd part of f : $f_e(x) = \frac{f(x) + \chi(x)f(\sigma(x))}{2}$, $f_o(x) = \frac{f(x) - \chi(x)f(\sigma(x))}{2}$ and $\chi f \circ \sigma$ satisfy (1.8) with g unchanged.

(ix) $f_e = f(e)g$. In particular $f_e = 0$ (f is odd) if and only if $f(e) = 0$

(x) The odd part f_o of f satisfies

$$(2.17) \quad f_o(xy) + f_o(yx) = 2f_o(x)g(y) + 2f_o(y)g(x)$$

for all $x, y \in G$.

(3) For the rest we assume that $\sigma(x) = x^{-1}$ for all $x \in G$.

(i) $\check{m}_g(G) \subseteq \{\pm 1\}$

If $m_g = \chi$, then either

(i) g is non-abelian and $f = f(e)g$, or

(iii) f and g are both abelian, in which case (1) applies.

(4) If $m_g \neq \chi$, then $f = -\chi f \circ \sigma$ (f is odd).

3. SOLUTIONS AND STABILITY OF THE FUNCTIONAL EQUATION (1.8), WHERE σ IS AN INVOLUTIVE HOMOMORPHISM OF G

In this section σ is an involutive homomorphism of G , that is $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$, for all $x, y \in G$. In the following theorem, we obtain the solutions of the functional equation (1.8) on semigroups with identity element. It turns out that, like on abelian groups, only multiplicative and additive functions occur in the solution formulas.

Theorem 3.1. *Let G be a semigroup with identity element, $\sigma: G \longrightarrow G$ a multiplicative function such that $\sigma \circ \sigma = I$, where I denotes the identity map, and $\chi: G \longrightarrow \mathbb{C}$ be a character of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$.*

The solutions f, g of the functional equation (1.8) are the following pairs of functions, where $m: G \longrightarrow \mathbb{C}$ denotes a function multiplicative and $c \in \mathbb{C}^$.*

(i) $f = 0$ and g arbitrary.

(ii) $g = \frac{m + \chi m \circ \sigma}{2}$ and $f = f(e)g$.

(iii) $g = \frac{m+\chi m \circ \sigma}{2}$ and $f = (c + \frac{f(e)}{2})m - (c - \frac{f(e)}{2})m \circ \sigma$ with $(\chi - 1)m = (\chi - 1)m \circ \sigma$.

(iv) $g = m$ and $f = (a + f(e))m$, where $m = \chi m \circ \sigma$ and $a: G \rightarrow \mathbb{C}$ is an additive map which satisfies $m(a \circ \sigma + a) = 0$.

Proof. It is elementary to check that the cases stated in the Theorem define solutions, so it is left to show that any solution $f, g: G \rightarrow \mathbb{C}$ of (1.8) falls into one of these cases. We use in the proof similar Stetkær's computations [38]. Let $x, y, z \in G$. If we replace x by xy and y by z in (1.8) we get

$$(3.1) \quad f(xyz) + \chi(z)f(\sigma(z)xy) = 2f(xy)g(z).$$

On the other hand if we replace x by $\sigma(z)x$ in (1.8), we obtain

$$\begin{aligned} f(\sigma(z)xy) + \chi(y)f(\sigma(y)\sigma(z)x) &= 2f(\sigma(z)x)g(y) \\ &= 2g(y)[\chi(\sigma(z))[2f(x)g(z) - f(xz)]]. \end{aligned}$$

Since,

$$\begin{aligned} \chi(y)f(\sigma(y)\sigma(z)x) &= \chi(y)f(\sigma(yz)x) = \chi(y)\chi(\sigma(yz))[2g(yz)f(x) - f(xyz)] \\ &= \chi(\sigma(z))[2g(yz)f(x) - f(xyz)], \end{aligned}$$

so by using $\chi(z\sigma(z)) = 1$ we have

$$(3.2) \quad \chi(z)f(\sigma(z)xy) + [2g(yz)f(x) - f(xyz)] = 2g(y)[2f(x)g(z) - f(xz)].$$

Subtracting this from (3.1) we get

$$(3.3) \quad f(xyz) = g(yz)f(x) + f(xy)g(z) + g(y)f(xz) - 2g(y)f(x)g(z).$$

With the notation

$$(3.4) \quad f_x(y) = f(xy) - f(x)g(y)$$

equation (3.3) can be written as follows

$$(3.5) \quad f_a(xy) = f_a(x)g(y) + f_a(y)g(x), \quad x, y \in G.$$

We will in the rest of the proof of Theorem 3.1 need to know the solutions of the functional equation

$$(3.6) \quad f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)f(y); \quad x, y \in G.$$

They are obtained in the following lemma.

Lemma 3.2. *Let G be a semigroup with identity element, $\sigma: G \rightarrow G$ a multiplicative function such that $\sigma \circ \sigma = I$, where I denotes the identity map, and $\chi: G \rightarrow \mathbb{C}$ be a character of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. The solutions f of the functional equation (3.6) are of the form $f = \frac{m+\chi m \circ \sigma}{2}$, where $m: G \rightarrow \mathbb{C}$ is multiplicative.*

Proof. Verifying that $f = \frac{m+\chi m \circ \sigma}{2}$, where $m: G \rightarrow \mathbb{C}$ is multiplicative, is solution of equation (3.6) consists in simple computations. Let f satisfies the functional equation (3.6), then by using the above computations the pair f, f_a satisfies equation

$$(3.7) \quad f_a(xy) = f_a(x)f(y) + f_a(y)f(x), \quad x, y \in G.$$

If $f_a = 0$ for all $a \in G$ then f is multiplicative. Substituting f in (3.6) we get $\chi(y)f(\sigma(y)) = f(y)$ for all $y \in G$. This implies that $f = \frac{\varphi+\chi\varphi}{2}$, where $f = \varphi$ is multiplicative.

If $f_a \neq 0$ for some $a \in G$. From the known solution of the sine addition formula (see for example [[40], Theorem 4.1]) there exist two multiplicative functions $\chi_1, \chi_2: G \rightarrow \mathbb{C}$ such that $f = \frac{\chi_1+\chi_2}{2}$. We can assume that $\chi_1 \neq \chi_2$. Substituting $f = \frac{\chi_1+\chi_2}{2}$ in (3.6) we get after reduction that

$$\chi_1(x)[\chi(y)\chi_1(\sigma(y)) - \chi_2(y)] = \chi_2(x)[\chi_1(y) - \chi(y)\chi_2(\sigma(y))].$$

Since $\chi_1 \neq \chi_2$ at least one of χ_1 and χ_2 is not zero. So, we get $\chi_1 = \chi\chi_2 \circ \sigma$, and $f = \frac{\varphi+\chi\varphi \circ \sigma}{2}$, where $\varphi: G \rightarrow \mathbb{C}$ is multiplicative. This completely describes the solutions of equation (3.6). \square

Now, we will find the solutions of equation (1.8). Let $f, g: G \rightarrow \mathbb{C}$ solution of equation (1.8). The above computation show that the pair f_a, g satisfies the sine addition formula (3.5) for any $a \in G$. From the known solution of the sine addition formula (see for example [[40], Theorem 4.1]) we have the following possibilities.

If $f = 0$ we deal with case (i). So during the rest of the proof we will assume that $f \neq 0$. If we replace a by e in (3.4) we get $f_e(x) = f(x) - f(e)g(x)$. If $f_e = 0$, then $f(x) = f(e)g(x)$ for all $x \in G$. Since $f \neq 0$ then $f(e) \neq 0$. Substituting $f = f(e)g$ into (1.8) we find that g satisfies equation (3.6) then there exists $m: G \rightarrow \mathbb{C}$ multiplicative such that $g = \frac{m+\chi m \circ \sigma}{2}$. We see that we deal with case (ii).

If $f_e \neq 0$, the pair (f_e, f) satisfies (3.5) and we known from [[40], Theorem 4.1] that there are only the following 3 possibilities:

(1) $f_e = cm$ and $g = m/2$ for some m multiplicative. Here $f = f_e + f(e)g$. Substituting $f = (c + \frac{f(e)}{2})m$, $g = m/2$ into (1.8) we find $(c + \frac{f(e)}{2})\chi(y)m(x)m(\sigma(y)) = 0$ for all $x, y \in G$. This case does not apply, because $f \neq 0$.

(2) There exist two different characters m and M on G and a constant $c \in \mathbb{C}^*$ such that

$$g = \frac{m+M}{2} \text{ and } f_e = c(m-M)$$

then $f = c(m-M) + f(e)\frac{m+M}{2} = \alpha m - \beta M$, where $\alpha = c + \frac{f(e)}{2}$ and $\beta = c - \frac{f(e)}{2}$. Substituting this into (1.8) we find after reduction that

$$(3.8) \quad \alpha m(x)(\chi(y)m(\sigma(y)) - M(y)) = \beta M(x)(\chi(y)M(\sigma(y)) - m(y)).$$

If we replace y by $\sigma(y)$ in (3.8) and after we multiply equation obtained by $\chi(y)$ and using $\chi(y\sigma(y)) = 1$ we find

$$(3.9) \quad \alpha m(x)(m(y) - \chi(y)M(\sigma(y))) = \beta M(x)(M(y) - \chi(y)m(\sigma(y))).$$

Subtracting (3.8) from (3.9) we get after some simplifications that

$$(3.10) \quad (\alpha m(x) + \beta M(x))(\chi(y)m(\sigma(y)) - M(y)) = (\alpha m(x) + \beta M(x))(\chi(y)M(\sigma(y)) - m(y)).$$

Putting $x = e$ in (3.10) we find that $\chi(y)m(\sigma(y)) - M(y) = \chi(y)M(\sigma(y)) - m(y)$, because $\alpha + \beta = 2c \neq 0$. If $\chi M \circ \sigma - m \neq 0$, then from (3.8) we get $\alpha m(x) = \beta M(x)$ for all $x \in G$. So, for $x = e$ we obtain $\alpha = \beta$ which contradicts the assumption that $f(e) \neq 0$. Thus, $M = \chi m \circ \sigma$ and $m = \chi M \circ \sigma$ from which we see that $g = \frac{m + \chi m \circ \sigma}{2}$ and $f = (c + \frac{f(e)}{2})m - (c - \frac{f(e)}{2})m \circ \sigma$. We conclude that we deal with case (iii).

(3) $g = m$ and $f_e = ma$, where m is multiplicative of G and a is an additive map. From $f_e = f - f(e)g$ we get $f = ma + f(e)m = (a + f(e))m$. Substituting this into (1.8) we find after reduction that

$$(3.11) \quad m(x)(a(y)m(y) + \chi(y)a(\sigma(y))m(\sigma(y))) + m(x)(a(x) + f(e))(\chi(y)m(\sigma(y)) - m(y)) = 0.$$

If we replace y by $\sigma(y)$ in (3.11) and after we multiply equation obtained by $\chi(y)$ and using $\chi(y\sigma(y)) = 1$ we find

$$(3.12) \quad m(x)(\chi(y)a(\sigma(y))m(\sigma(y)) + a(y)m(y)) + m(x)(a(x) + f(e))(m(y) - \chi(y)m(\sigma(y))) = 0.$$

Subtracting (3.11) from (3.12) we get after some simplifications that

$$(3.13) \quad 2m(x)(a(x) + f(e))(\chi(y)m(\sigma(y)) - m(y)) = 0$$

for all $x, y \in G$. Putting $x = e$ in (3.13) we get $m = \chi m \circ \sigma$, because $2m(e)(a(e) + f(e)) = 2.1.(0 + f(e)) = 2f(e) \neq 0$. This means that $g = \frac{m + \chi m \circ \sigma}{2}$. Substituting $m = \chi m \circ \sigma$ into (3.11) we deduce that $m(a \circ \sigma + a) = 0$. We see that we deal with case (iv) and this completes the proof. \square

The formulas of Theorem 3.1 implies the following corollary.

Corollary 3.3. *Let G be a semigroup with identity element, $\sigma: G \rightarrow G$ a multiplicative function such that $\sigma \circ \sigma = I$, where I denotes the identity map, and $\chi: G \rightarrow \mathbb{C}$ be a multiplicative function of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. If $f, g: G \rightarrow \mathbb{C}$ is a solution of variant of Wilson's functional equation (1.8) such that $f \neq 0$, then g is a solution of variant of d'Alembert's functional equation (3.6).*

In the rest of this section we examine the Hyers-Ulam stability of the functional equation (1.8). We shall first recall two variants of Székelyhidi results because it will be useful in the treatment of stability of other functional equations like sine addition formula. The proof of Theorem 3.3 and Theorem 3.4 goes along the same lines as the one in [46] and [47].

Theorem 3.4. [46] *Let V be a vector space of \mathbb{C} -valued functions on a semigroup G , let V be left invariant and suppose that f and m are \mathbb{C} -valued functions on G . If the function $y \mapsto f(xy) - f(y)m(x)$ belongs to V for all $x \in G$. Then either f is in V or m is an exponential.*

Theorem 3.5. [49] *Let V be a vector space of \mathbb{C} -valued functions on a semigroup G , let V be invariant and suppose that f and g are \mathbb{C} -valued functions on G which are linearly independent modulo V . If the function $x \mapsto f(yx) - f(x)g(y) - f(y)g(x)$ belongs to V for all $y \in G$, then $f(xy) = f(x)g(y) + f(y)g(x)$ for all $x, y \in G$.*

In the following theorem we obtain the Hyers-Ulam stability of the functional equation (1.8). The following lemmas will be helpful in the sequel.

Lemma 3.6. *Let $\delta \geq 0$, let G be a semigroup with identity element, $\sigma: G \rightarrow G$ is an homomorphism such that $\sigma \circ \sigma = I$, and $\chi: G \rightarrow \mathbb{C}$ be a bounded multiplicative function such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair $f, g: G \rightarrow \mathbb{C}$ satisfies*

$$(3.14) \quad |f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)| \leq \delta, \text{ for all } x, y \in G.$$

Under these assumptions the following statement hold:

$$(3.15) \quad |f_a(xy) - f_a(x)g(y) - f_a(y)g(x)| \leq |g(x)|\delta + \frac{3}{2}\delta, \text{ for all } x, y \in G,$$

where f_a is the function defined in (3.4).

Proof. For $x, y \in G$ we put $F(x, y) = f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)$. By using similar computations used in the proof of Theorem 3.1 we get

$$(3.16) \quad \begin{aligned} & f(xyz) - f(x)g(yz) + 2f(x)g(y)g(z) - f(xy)g(z) - g(y)f(xz) \\ &= -g(y)F(x, z) + \frac{F(x, yz)}{2} + \frac{F(xy, z)}{2} - \frac{F(\sigma(z)x, y)}{2} \end{aligned}$$

for all $x, y, z \in G$. From inequality (3.14) and the definition of f_a we get the desired result. \square

The second main result of this section is the next one.

Theorem 3.7. *Let G be a group with identity element, $\sigma: G \rightarrow G$ an involutive homomorphism of G and $\chi: G \rightarrow \mathbb{C}$ be a unitary character of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Let the pair $f, g: G \rightarrow \mathbb{C}$ be given. Suppose that the function*

$$(3.17) \quad (x, y) \mapsto f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)$$

is bounded. Under these assumptions the following statements hold:

- (i) $f = 0$ and g arbitrary.
- (ii) $f \neq 0$ is bounded and g is bounded.
- (iii) f is unbounded, g is bounded and G is an amenable group, then $g \neq 0$ is multiplicative, $g = \chi g \circ \sigma$ and there exists an additive map $a: G \rightarrow \mathbb{C}$ such that $f - ag$ is bounded and $(ag)(xy) + \chi(y)(ag)(\sigma(y)x) = 2(ag)(x)g(y)$

for all $x, y \in G$.

(iv) f is unbounded, g is unbounded. In this case there are the following three possibilities:

(1) g is multiplicative, $g = \chi g \circ \sigma$, $f = f(e)g$. Furthermore, f, g satisfy equation (1.8).

(2) g is multiplicative, $g = \chi g \circ \sigma$, $f = ag$, where a is an additive map such that $a \circ \sigma = -a$. Furthermore, f, g satisfy equation (1.8).

(3) $g = \frac{m + \chi m \circ \sigma}{2}$ and $f = (c + \frac{f(e)}{2})m - (c - \frac{f(e)}{2})m \circ \sigma$, where m is multiplicative.

Proof. If $f = 0$ we deal with case (i). So during the rest of the proof we will assume that $f \neq 0$. If f is bounded then by using (3.17) we get g bounded. This is case (ii)

(iii) If f is unbounded and g bounded. We notice here that $g \neq 0$, because if $g = 0$ then from (3.17) with $y = e$ we get f bounded, which contradict our assumption that f is unbounded. We put $h = f - f(e)$, so $h(e) = 0$ and the function

$$(3.18) \quad (x, y) \longrightarrow h(xy) - h(x)g(y) - h(y)g(x)$$

is bounded. Thus, the function $y \mapsto h(xy) - h(y)g(x)$ is bounded for all $x \in G$. So, by using Theorem 3.4 we get $g = m$ multiplicative and the function defined in (3.18) remains bounded when the right side is multilied by $m((xy)^{-1}) = m(x^{-1})m(y^{-1})$, so that the function $(x, y) \longrightarrow \frac{h}{m}(xy) - \frac{h}{m}(x) - \frac{h}{m}(y)$ is bounded. Since G is amenable then from [50] we have $\frac{h}{m}(x) = a(x) + b(x)$ for all $x \in G$, where $a: G \rightarrow \mathbb{C}$ is an unbounded additive map and $b: G \rightarrow \mathbb{C}$ is bounded. On the other hand by Substituting this into $B(x, y) = h(xy) + \chi(y)h(\sigma(y)x) - 2h(x)g(y)$ we get $a(xy)m(xy) + b(xy)m(xy) + \chi(y)[a(\sigma(y)x)m(\sigma(y)x) + b(\sigma(y)x)m(\sigma(y)x)] = 2[a(x)m(x) + b(x)m(x)]m(y) + B(x, y)$ and we find after reduction that the function

$$(3.19) \quad |a(x)m(x)(\chi(y)m(\sigma(y)) - m(y)) + a(y)m(x)m(y) + \chi(y)a(\sigma(y))m(x)m(\sigma(y))| \leq \delta$$

for all $x, y \in G$ and for some $\delta \geq 0$. By replacing y by $\sigma(y)$ in (3.19) and using $\chi(y\sigma(y)) = 1$ we get

$$(3.20) \quad |a(x)m(x)(m(y) - \chi(y)m(\sigma(y))) + \chi(y)a(\sigma(y))m(x)m(\sigma(y)) + a(y)m(x)m(y)| \leq \delta.$$

Subtracting (3.19) from (3.20) we get after some simplifications that

$$(3.21) \quad |2a(x)m(x)| |m(y) - \chi(y)m(\sigma(y))| \leq 2\delta$$

for all $x, y \in G$. Since $|m(x)| = 1$ and a is unbounded then we get $m(y) = \chi(y)m(\sigma(y))$ for all $y \in G$.

Now, we will show that $l = ag$ satisfies $l(xy) + \chi(y)l(\sigma(y)x) = 2l(x)m(y)$.

For all $x, y \in G$ we have

$$\begin{aligned} & l(xy) + \chi(y)l(\sigma(y)x) - 2l(x)m(y) \\ &= (a(x) + a(y))m(x)m(y) + \chi(y)(a(\sigma(y)) + a(x))m(\sigma(y))m(x) - 2a(x)m(x)m(y) \end{aligned}$$

$$= (a(y) + a(\sigma(y)))m(x)m(y).$$

Since

$$(3.22) \quad |h(xy) + \chi(y)h(\sigma(y)) - 2h(x)g(y)| \leq \beta$$

for some $\beta \geq 0$, $h(e) = 0$, $h = m(a + b)$, m is multiplicative, $|m(x)| = 1$, $m = \chi m \circ \sigma$ and b is bounded then if we put $x = e$ in (3.22) we get

$$|m(y)a(y) + m(y)b(y) + \chi(y)m(\sigma(y))a(\sigma(y)) + \chi(y)m(\sigma(y))b(\sigma(y))| \leq \beta.$$

This means that the function $y \rightarrow |a(y) + a(\sigma(y))|$ is bounded. Since $a + a \circ \sigma$ is an additive map then we get $a(y) + a(\sigma(y)) = 0$ for all $y \in G$ and we conclude that $l(xy) + \chi(y)l(\sigma(y)x) = 2l(x)g(y)$ for all $x, y \in G$, and we see that we deal with case (iii).

If f, g are unbounded, then by using (3.15) we get that either $f_a = 0$ for all $a \in G$ or f_a is unbounded for all $a \in G$. Indeed, if there exists $a \in G$ with $f_a \neq 0$ and f_a bounded, so from inequality (3.15) with $x = x_0$ where $f_a(x_0) \neq 0$ we get g bounded which contradicts the assumption that g is unbounded.

In this case we have the following possibilities:

If $f_a = 0$ for all $a \in G$ then $f(xy) = f(x)g(y)$ for all $x, y \in G$ and this implies that g is multiplicative and $f = f(e)g$. Substituting this into (3.17) we get after reduction that $|g(x)|\chi(y)g(\sigma(y)) - g(y)| \leq \gamma$ for all $x, y \in G$ and for some $\gamma \geq 0$. Since g is unbounded we deduce that $g = \chi g \circ \sigma$. So, g satisfies equation (3.9), and the pair f, g satisfies equation (1.8). We deal with case (iv)(1).

If there exists $a \in G$ such that $f_a \neq 0$, then by using the above notice we get f_a is unbounded for all $a \in G$. For the rest of the proof we put $a = e$ and we will discuss two cases.

First Case: If f_e, g are linearly dependent modulo the spaces of complex bounded function on G (see [49]), then there exists a constant $\lambda \in \mathbb{C}^*$ and a bounded function b on G such that $g = \frac{1}{2\lambda}f_e + b$. Substituting this into inequality (3.15) we get

$$|f_e(xy) - f_e(x)[\frac{1}{2\lambda}f_e(y) + b(y)] - f_e(y)[\frac{1}{2\lambda}f_e(x) + b(x)]| \leq |\frac{1}{2\lambda}f_e(x) + b(x)|\delta + \frac{3}{2}\delta$$

for all $x, y \in G$, so we have

$$|f_e(xy) - (\frac{1}{\lambda}f_e(x) + b(x))f_e(y)| \leq |f_e(x)||b(y)| + |\frac{1}{2\lambda}f_e(x) + b(x)|\delta + \frac{3}{2}\delta.$$

Thus the function $y \rightarrow f_e(xy) - (\frac{1}{\lambda}f_e(x) + b(x))f_e(y)$ is bounded for all $x \in G$. Since f_e is unbounded then from Theorem 3.4 (with V is the space of bounded function on G) we get $m = \frac{1}{\lambda}f_e + b$ multiplicative, $f_e = \lambda m - \lambda b$, $g = \frac{m}{2} + \frac{b}{2}$ and $f = f_e + f(e)g = (\lambda + \frac{f(e)}{2})m + (\frac{f(e)}{2} - \lambda)b = \alpha m + \beta b$. Substituting this into bounded function $B(x, y) = f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)$ we find after reduction that

$$(3.23) \quad \alpha m(x)[\chi(y)m(\sigma(y)) - b(y)] = \beta b(x)m(y) + \beta b(x)b(y) - \beta \chi(y)b(\sigma(y)x) + B(x, y)$$

for all $x, y \in G$. Since b is bounded, m is unbounded and $|\chi(y)| = 1$ then there exists $a \in G$ such that $\chi(a)m(\sigma(a)) - b(a) \neq 0$. From (3.23) we conclude that m is a bounded multiplicative and this case does not apply, because m is unbounded. So we have the second case:

Case 2: f_e, g are linearly independent modulo the spaces of complex bounded function on G . From inequality (3.15) and Theorem 3.5, (with V is the space of bounded function on G) reveals that the pair (f_e, g) is a solution of the sine addition formulas

$$(3.24) \quad f_e(xy) = f_e(x)g(y) + f_e(y)g(x)$$

for all $x, y \in G$, so we known from [[40], Corollary 4.4] that there are only the following possibilities:

(1) $f_e = cm$ and $g = \frac{m}{2}$ for some multiplicative function $m : G \rightarrow \mathbb{C}$. Here $f = f_e + f(e)g = (c + \frac{f_e}{2})m = \gamma m$. Substituting this into bounded function $B(x, y) = f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)$ we find after reduction that $\beta\chi(y)m(\sigma(y))m(x) = B(x, y)$. This means that m is a bounded multiplicative and this case does not apply, because m is unbounded.

(2) $g = m$ and $f_e = am$ for some multiplicative function $m : G \rightarrow \mathbb{C}$ and $a : G \rightarrow \mathbb{C}$ an additive map. In this case $f = (a + f(e))m$, so we find after reduction that the bounded function:

$$(3.25) \quad \begin{aligned} & f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y) = (a(x) + a(y) + f(e))m(x)m(y) \\ & + \chi(y)[(a(\sigma(y)) + a(x) + f(e))m(\sigma(y))m(x) - 2(a(x) + f(e))m(x)m(y)] \\ & = (a(x) + f(e))m(x)(\chi(y)m(\sigma(y)) - m(y)) + m(x)[\chi(y)a(\sigma(y))m(\sigma(y)) + a(y)m(y)]. \end{aligned}$$

If we replace y with $\sigma(y)$ in (3.25), and after we multiply equation obtained by $\chi(y)$ and using $\chi(y\sigma(y)) = 1$ we get

$$(3.26) \quad (a(x) + f(e))m(x)(m(y) - \chi(y)m(\sigma(y))) + m(x)(a(y)m(y) + \chi(y)a(\sigma(y))m(\sigma(y)))$$

which is also a bounded function. Subtracting (3.25) from (3.26) we get after some simplifications that the function $(x, y) \mapsto m(x)(a(x) + f(e))(m(y) - \chi(y)m(\sigma(y)))$ is bounded. Since $f = m(a + f(e))$ is unbounded then we get $m = \chi m \circ \sigma$. Now, we will verify that the pair (f, g) is a solution of equation (1.8). for all $x, y \in G$ we have

$$\begin{aligned} & f(xy) + \chi(y)f(\sigma(y)x) - f(x)g(y) \\ & = (a(x) + a(y) + f(e))m(x)m(y) \\ & + \chi(y)[a(\sigma(y)) + a(x) + f(e))m(\sigma(y))m(x)] - 2(a(x) + f(e))m(x)m(y) \\ & = (a(y) + a(\sigma(y)))m(x)m(y). \end{aligned}$$

Since $(x, y) \mapsto f(xy) + \chi(y)f(\sigma(y)x) - f(x)g(y)$ is a bounded function, then we have $(x, y) \mapsto (a(y) + a(\sigma(y)))m(x)m(y)$ is also bounded. Since m is unbounded then we get the desired result, so we see that we deal with case (iv) (2).

(3) There exit two different characters m, M and a constant $c \in \mathbb{C}^*$ such that

$g = \frac{m+M}{2}$ and $f_e = c(m - M)$. In this case $f = f_e + f(e)g = (c + \frac{f(e)}{2})m - (c - \frac{f(e)}{2})M = \alpha m - \beta M$, where $\alpha = c + \frac{f(e)}{2}$ and $\beta = c - \frac{f(e)}{2}$. Substituting this into bounded function $B(x, y) = f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y)$ we find after reduction that

$$(3.27) \quad \alpha m(x)(\chi(y)m(\sigma(y)) - M(y)) + \beta M(x)(m(y) - \chi(y)M(\sigma(y))) = B(x, y).$$

If we replace y with $\sigma(y)$ in (3.27), and after we multiply equation obtained by $\chi(y)$ and using $\chi(y\sigma(y)) = 1$ we get

$$(3.28) \quad \alpha m(x)(m(y) - \chi(y)M(\sigma(y))) + \beta M(x)(\chi(y)m(\sigma(y)) - M(y)) = \chi(y)B(x, \sigma(y)).$$

If we add (3.27) to (3.28) we get after some simplifications that the function $(x, y) \mapsto (\alpha m(x) + \beta M(x))[(m(y) - \chi(y)M(\sigma(y))) + (\chi(y)m(\sigma(y)) - M(y))]$ is bounded. Since $\alpha m + \beta M = 2cg + \frac{f(e)}{c}f_e$ and g, f_e are linearly independent modulo the space of complex bounded functions on G , $\alpha m + \beta M = 2cg + \frac{f(e)}{c}f_e$ is unbounded then we get $m - \chi M \circ \sigma = M - \chi m \circ \sigma$. Now, the bounded function (3.27) can be written as follows

$$(3.29) \quad \begin{aligned} f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)g(y) &= (\alpha m(x) - \beta M(x))(\chi(y)m(\sigma(y)) - M(y)) \\ &= f(x)(\chi(y)m(\sigma(y)) - M(y)) \end{aligned}$$

Since f is assumed to be unbounded then we get $\chi(y)m(\sigma(y)) = M(y)$ for all $y \in G$ and g take the expression: $g = \frac{m + \chi m \circ \sigma}{2}$. Equation (3.29) show that the pair (f, g) satisfies equation (1.8). We see that we deal with case (iv) (3) and this completes the proof. \square

As an application we get the superstability of the functional equation (3.6).

Corollary 3.8. *Let $\delta \geq 0$. Let G be a group with identity element, $\sigma: G \rightarrow G$ an involutive homomorphism and $\chi: G \rightarrow \mathbb{C}$ be a unitary character of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Let $f: G \rightarrow \mathbb{C}$ such that*

$$(3.30) \quad |f(xy) + \chi(y)f(\sigma(y)x) - 2f(x)f(y)| \leq \delta$$

for all $x, y \in G$. Then either f is bounded or f satisfies equation (3.6)

In [36], the authors presented some rich ideas on the study of the superstability of symmetrized multiplicative Cauchy equation

$$(3.31) \quad f(xy) + f(yx) = 2f(x)f(y) \quad x, y \in G.$$

However, we have formulate the problem as an open problem. The solutions of equation (3.31) are multiplicative functions (see for exapmle [43]). In the following, we give the affirmative answer. If we put $\chi = 1$ and $\sigma = I$ in Corollary 3.8, where I denotes the identity map we get

Corollary 3.9. *Let $\delta \geq 0$. Let G be a group with identity element. Let $f : G \rightarrow \mathbb{C}$ such that*

$$(3.32) \quad |f(xy) + f(yx) - 2f(x)f(y)| \leq \delta$$

for all $x, y \in G$. Then either f is bounded or f is multiplicative.

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Elqorachi Elhoucien, Department of Mathematics, Faculty of Sciences, University Ibn Zohr, Agadir, Morocco,

E-mail: elqorachi@hotmail.com

Redouani Ahmed, Department of Mathematics, Faculty of Sciences, University Ibn Zohr, Agadir, Morocco,

E-mail: Redouani-ahmed@yahoo.fr